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1992 J. Phys. A: Math. Gen. 25 2521

(http://iopscience.iop.org/0305-4470/25/9/023)

J. Phys. A: Math. Gen. 25 (1992) 2521–2531. Printed in the UK

Thermal transmissivity in discrete spin systems: formulation and applications

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Received 14 November 1991

Abstract. Thermal transmissivity, a notion which arises empirically and leads to considerable simplification of analyses of discrete spin systems, is of fundamental interest in graphical and real-space renormalization group approaches. In this paper we present a formulation of the theory of thermal transmissivity. We show that the situation is different depending on whether the interaction matrices commute. In the commuting case the thermal transmissivity is given by the eigenvalues of the interaction matrix, while in the non-commuting case it is given by the eigenvalues as well as matrices which block-diagonalize the interaction matrix. The meaning of the thermal transmissivity in diagrammatic analyses of spin models is also elucidated, and our results are illustrated by examples. Finally, we present a general formulation of disorder solutions for spin models in terms of thermal transmissivities.

1. Introduction

In the analyses of discrete spin systems certain variables arise empirically which lead to considerable simplification. A well-known example is the hyperbolic tangent function occurring in high-temperature expansions of the Ising model [1]. Let K denote the Ising interaction, then a sequence of Ising interactions $K_1, K_2, \ldots$ can be replaced by a single K given by the product relation $\tanh K = \prod_i \tanh K_i$. Furthermore, in the diagrammatic expansion of the partition function using $\tanh K$ as a bond variable, one finds terms containing vertices with an odd number of incident bonds vanish identically. Because of its usefulness in decimations of spins as well as in real-space renormalization group treatments, this variable has been coined the term thermal transmissivity, or simply *transmissivity* [2]. Explicit forms of transmissivity have also been obtained for other spin systems, including the Potts model [3, 4], the $Z(q)$ model [5] and, very recently, the discrete cubic models [6, 7]. In view of its fundamental importance, we present a general formulation of thermal transmissivity for any spin system. Our results are illustrated with various examples.

Consider a general q-state spin system for which two spins interact via a set of interaction parameters $\mathsf{K} \equiv \{K_1, K_2, \ldots\}$. Generally, any two-spin interaction can be characterized by an interaction matrix $\mathsf{W}(\mathsf{K})$ whose element $W_{\alpha\beta}(\mathsf{K})$ is the Boltzmann factor between two spin states α and $\beta = 0, 1, \ldots, q-1$. In this paper

0305-4470/92/092521+11$04.50

we confine ourselves to (stochastic) interaction matrices with each row and column containing the same set of Boltzmann factors. Thus we have

$$\sum_{\beta=0}^{q-1} W_{\alpha\beta}(\mathbf{K}) = \sum_{\alpha=0}^{q-1} W_{\alpha\beta}(\mathbf{K}) = \lambda_0 \tag{1}$$

where λ_0 is independent of α, and the matrix $\mathbf{W}$ is not necessarily symmetric. Practically all spin models of physical interests are of this kind including, among others, the Ising, Potts [8], chiral Potts [9, 10], Ashkin-Teller [11], $\{N_\alpha, N_\beta\}$ [12], $Z(q)$ and cubic [13] models. We then look for entities, which can be either scalar functions $t_i(\mathbf{K})$ and/or matrices $\mathbf{T}_i(\mathbf{K})$, such that the decimation of the intervening spin in a sequence of two interactions $\mathbf{K}$, $\mathbf{K}'$ yields a set of effective interactions $\mathbf{K}''$ given by the product relation

$$t_i(\mathbf{K}'') = t_i(\mathbf{K})t_i(\mathbf{K}') \qquad \mathbf{T}_i(\mathbf{K}'') = \mathbf{T}_i(\mathbf{K})\mathbf{T}_i(\mathbf{K}'). \tag{2}$$

The role of these entities in diagrammatic analyses will also be examined.

2. Commuting interaction matrices

We consider first the case of commuting and diagonalizable interaction matrices, i.e. interaction matrices with different parameters commute and can be simultaneously diagonalized. Physically, the commutation of interaction matrices means that the physics is unchanged if two interactions connected in series are interchanged; this is often the case in models of physical interest. Then, except in the exotic cases of nilpotent matrices, the interaction matrices with different parameters can be simultaneously diagonalized. That is, there exists a non-singular $q \times q$ matrix $\mathbf{P}$, *independent* of $\mathbf{K}$, such that

$$\mathbf{PW(K)P}^{-1} = \begin{pmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{q-1} \end{pmatrix} \tag{4}$$

where $\lambda_i = \lambda_i(\mathbf{K})$ are the eigenvalues of $\mathbf{W(K)}$ and λ_0 is the largest one which is given by (1) by Frobenius' theorem.

Now the interaction matrix for a series of two interactions $\mathbf{K}$ and $\mathbf{K}'$ is, by definition, $\mathbf{W(K)W(K')}$. Clearly, this matrix is also diagonalized by the similarity transformation (4) with diagonal elements $\lambda_i(\mathbf{K})\lambda_i(\mathbf{K}')$. This points to the entities

$$t_i(\mathbf{K}) = \lambda_i(\mathbf{K})/\lambda_0(\mathbf{K}) \tag{5}$$

with $i \neq 0$ ranging over all distinct eigenvalues, possessing the product property (2), and therefore can be taken to be the thermal transmissivities. Indeed, for the zero-field Ising model we have $\lambda_0 = 2\cosh K$, $\lambda_1 = 2\sinh K$, $t(K) = \lambda_1/\lambda_0 = \tanh K$. Other examples are given in section 5. It should be pointed out that if the interaction matrices are not diagonalizable, but can be simultaneously transformed into a triangular form with the eigenvalues appearing in the main diagonal, then the transmissivities can also be taken to be (5).

3. Diagrammatic analyses

We continue discussions of the case of commuting interaction matrices and examine the role played by the transmissivity (5) in diagrammatic analyses. Let $\mathbf{u}_i$ (a column matrix) and $\mathbf{v}_i$ (a row matrix) be the respective right and left eigenvector corresponding to λ_i, which are shared by the commuting interaction matrices, satisfying†

$$\mathbf{W}\mathbf{u}_i = \lambda_i \mathbf{u}_i \qquad \mathbf{v}_i\mathbf{W} = \lambda_i \mathbf{v}_i \qquad \mathbf{v}_i\mathbf{u}_j = \delta_{ij}. \tag{6}$$

By Frobenius' theorem and explicit construction all elements of $\mathbf{u}_0$ and $\mathbf{v}_0$ are equal to $1/\sqrt{q}$. Define $q \times q$ matrices

$$\mathbf{M}_i \equiv \mathbf{u}_i\mathbf{v}_i \qquad i = 0, 1, \ldots, q-1. \tag{7}$$

We have

$$\mathbf{M}_0 = \frac{1}{q}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \tag{8}$$

and, using (6),

$$\mathbf{M}_i\mathbf{M}_j = \mathbf{M}_i\delta_{ij} \tag{9}$$

and thus we can write

$$\mathbf{W} = \lambda_0\left[\mathbf{M}_0 + \frac{1}{q}\mathbf{F}\right] \tag{10}$$

where

$$\mathbf{F} = q\sum_{i=1}^{q-1} t_i \mathbf{M}_i. \tag{11}$$

Explicitly, we have

$$F_{\alpha\beta} = q\frac{W_{\alpha\beta}}{\lambda_0} - 1 \tag{12}$$

and, from (1),

$$\sum_{\beta=0}^{q-1} F_{\alpha\beta} = 0. \tag{13}$$

† It is assumed that, in the case of degenerate eigenvalues, we have formed linear combinations of the eigenvectors so that (6) holds.

Thus, if $F_{\alpha\beta}$ is taken to be the bond variable in diagrammatic expansions, all diagrams containing vertices with a single incident bond will have zero weight. This fact considerably reduces the number of diagrams that need to be considered. Furthermore, using (6) and (9), one finds for a series of two interactions $\mathbf{K}$ and $\mathbf{K}'$,

$$\sum_{\beta=0}^{q-1} F_{\alpha\beta}(t_i)F_{\beta\gamma}(t_i') = F_{\alpha\gamma}(t_i t_i') \tag{15}$$

where $t_i = t_i(\mathbf{K}), t_i' = t_i(\mathbf{K}')$, $i(\neq 0)$ ranging over all transmissivities. Therefore, the intervening spins in a sequence of interactions can be conveniently decimated, with the net result that the sequence is replaced by a single interaction whose effective transmissivity is given by the product relation (2). This property has been found to be extremely useful in analysing spin models [4, 14] and in carrying out real-space renormalization group analyses [2, 3, 5–7].

4. Non-commuting interaction matrices

If different interaction matrices do not commute, they cannot be simultaneously diagonalized. However, by Frobenius' theorem the largest eigenvalue λ_0 is always given by (1) and non-degenerate, and therefore can be singled out by taking a similarity transformation $\mathbf{P}$ with $P_{0i} = (P^{-1})_{i0} = 1/\sqrt{q}$. Furthermore, there may exist some symmetry in the interaction which permits further simultaneous diagonalization and/or block-diagonalization of the remainder of the interaction matrices. Then, the transmissivity can be taken to be the set of eigenvalues and block matrices thus obtained. Of course, if one is only interested in transmissivities which are scalars, one can always take the associated determinants instead of the matrices themselves. In case some of the block matrices are triangular, then the set includes the diagonal elements of these triangular matrices, which are eigenvalues themselves, instead of the matrices.

An important class of spin models with non-commuting interaction matrices satisfying (1) are the interaction models introduced by Biggs [15, 16]. Let $\alpha, \beta, \ldots$ denote the elements of a group G of order N. Then the interaction model is an N-state spin model with Boltzmann weights

$$W_{\alpha\beta} = W(\alpha^{-1}\beta) \qquad \alpha, \beta \in G. \tag{16}$$

Here, the interaction matrices (16) are elements of the group algebra for G. Namely, the decimation of a spin in a sequence of two interactions gives rise to an effective interaction of the same type, a fact which follows from the identity

$$\begin{aligned}\sum_{\beta\in G} W_{\alpha\beta}W'_{\beta\gamma} &= \sum_{\beta\in G} W(\alpha^{-1}\beta)W'(\beta^{-1}\gamma)\\ &= \sum_{x\in G} W(x)W'(x^{-1}\alpha^{-1}\gamma) = W''(\alpha^{-1}\gamma) = W''_{\alpha\gamma}.\end{aligned} \tag{17}$$

We use the permutation group $G = S_3$ to illustrate our discussions. For definiteness order the six elements $\{e, P_{12}P_{23}, P_{23}P_{12}, P_{12}, P_{23}, P_{31}\}$ of S_3 in the order given, where e is the identity permutation and the P_{ij} the transpositions. The interaction matrix (16) then takes the form

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}_1 & \mathbf{W}_2 \\ \mathbf{W}_2 & \mathbf{W}_1 \end{pmatrix} \tag{18}$$

where

$$\mathbf{W}_1 = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \qquad \mathbf{W}_2 = \begin{pmatrix} d & e & f \\ e & f & d \\ f & d & e \end{pmatrix}. \tag{18a}$$

Note that $\mathbf{W}_2$ is not cyclic†. Using a similarity transformation generated by

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{q} & \mathbf{q} \\ \mathbf{q} & -\mathbf{q} \end{pmatrix} \tag{19}$$

we find

$$\mathbf{P}\mathbf{W}\mathbf{P}^{-1} = \begin{pmatrix} \mathbf{q}(\mathbf{W}_1 + \mathbf{W}_2)\mathbf{q}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{q}(\mathbf{W}_1 - \mathbf{W}_2)\mathbf{q}^{-1} \end{pmatrix}. \tag{20}$$

Further choosing

$$\mathbf{q} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \tag{21}$$

where $\omega = \mathrm{e}^{\mathrm{i}2\pi/3}$, we find

$$\begin{aligned} &\mathbf{q}(\mathbf{W}_1 \pm \mathbf{W}_2)\mathbf{q}^{-1} = \begin{pmatrix} \lambda_\pm & \mathbf{0} \\ \mathbf{0} & \mathbf{m}_\pm \end{pmatrix} \\ &\lambda_\pm = a + b + c \pm (d + e + f) \\ &\mathbf{m}_\pm = \begin{pmatrix} u_{11} & \pm u_{12} \\ \pm u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} a + b\omega^2 + c\omega & \pm(d + e\omega + f\omega^2) \\ \pm(d + e\omega^2 + f\omega) & a + b\omega + c\omega^2 \end{pmatrix}. \end{aligned} \tag{22}$$

Here $\lambda_+ = \lambda_0$ is the largest eigenvalue. In a similar fashion the product of two matrices $\mathbf{W}\mathbf{W}'$ is transformed by the same similarity transformation into one with block diagonal elements $\lambda_\pm \lambda'_\pm$ and $\mathbf{m}_\pm \mathbf{m}'_\pm$. The transmissivities are now taken to be the function t_1 and the two *matrices* given by

$$t_1 = \lambda_-/\lambda_+ \qquad \mathbf{T}_1 = \mathbf{m}_+/\lambda_+ \qquad \mathbf{T}_2 = \mathbf{m}_-/\lambda_+. \tag{23}$$

To clarify the meaning of the transmissivity (23), we consider the diagrammatic expansion (of the partition function, for example) and write $\mathbf{W}$ in the form of (10) but now with

$$\mathbf{F} = q[t_1 \mathbf{M}_1 + \mathbf{P}_1 \mathbf{T}_1 \mathbf{Q}_1 + \mathbf{P}_2 \mathbf{T}_2 \mathbf{Q}_2] \tag{24}$$

† With other orderings of elements of S_3 corresponding to the interchange of some columns and rows, $\mathbf{W}$ can be put into other forms including one given by $\begin{pmatrix} \mathbf{W}_1 & \mathbf{W}_2 \\ \tilde{\mathbf{W}}_2 & \mathbf{W}_1 \end{pmatrix}$ where both $\mathbf{W}_1$ and $\mathbf{W}_2$ are cyclic, and $\tilde{\mathbf{W}}_2$ denotes the transpose of $\mathbf{W}_2$.

where $\mathbf{M}_1$ is given by (7), $\mathbf{P}_i$ is the 6×2 matrix consisting of the $(3i-1)$th and $(3i)$th columns of $\mathbf{P}^{-1}$, and $\mathbf{Q}_i$ the 2×6 matrix consisting of the $(3i-1)$th and $(3i)$th rows of $\mathbf{P}$. Again, the identity (13) holds, meaning that all diagrams containing vertices with a single incident bond will have zero weight in the expansion. We further find, in analogy to (15),

$$\sum_{\beta=0}^{5} F_{\alpha\beta}(t_1,\mathbf{T}_1,\mathbf{T}_2)F_{\beta\gamma}(t_1',\mathbf{T}_1',\mathbf{T}_2') = F_{\alpha\gamma}(t_1t_1',\mathbf{T}_1\mathbf{T}_1',\mathbf{T}_2\mathbf{T}_2'). \tag{25}$$

Thus, the decimation of an intervening spin in a series of two interactions generates an equivalent transmissivity according to the product rule (2), now applying to t_1 as well as matrices $\mathbf{T}_1$ and $\mathbf{T}_2$.

While the two matrices $\mathbf{m}_\alpha = \mathbf{m}_\alpha(a,b,\ldots,f)$ and $\mathbf{m}'_\alpha = \mathbf{m}_\alpha(a',b',\ldots,f')$, $\alpha = +$ or $-$, with different parameters generally do not commute, it is readily verified that they do commute if we have

$$u_{12}/u_{21} = u'_{12}/u'_{21} \qquad (u_{11}-u_{22})/u_{12} = (u'_{11}-u'_{22})/u'_{12}. \tag{26}$$

That is, $\mathbf{m}_\pm$ and hence $\mathbf{T}_1$ and $\mathbf{T}_2$, can be completely diagonalized by a similarity transformation, independent of the parameters, in the parameter space

$$\frac{e-f}{2d-e-f} = C_1 \qquad \frac{(b-c)^2}{d^2+e^2+f^2-bd-ef-fd} = C_2 \tag{27}$$

where C_1 and C_2 are constants. In this subspace the transmissivities are given by $t_i = \lambda_i/\lambda_0$, $i = 1,2,\ldots,5$, where λ_i are the eigenvalues of $\mathbf{W}$. Explicit examples satisfying (27) are given in the next section.

5. Examples

Ising model. $t = \lambda_1/\lambda_0 = \tanh K$.
Potts model. The interaction matrix has two distinct eigenvalues, $\lambda_0 = \mathrm{e}^K + q - 1$ and a $(q-1)$-fold degenerate $\lambda_1 = \mathrm{e}^K - 1$. Hence, $t = \lambda_1/\lambda_0 = (\mathrm{e}^K-1)/(\mathrm{e}^K+q-1)$.
q-state chiral Potts model [9, 10]. Let the lattice edges be oriented from site i to site j such that the Boltzmann factor is $W_{\alpha_i\alpha_j} = W(\alpha_i-\alpha_j), (\mathrm{mod}\, q)$. Then, there exist $q-1$ distinct transmissivities

$$t_\beta(\mathbf{W}) = \frac{1}{\lambda_0}\sum_{\alpha=0}^{q-1} \mathrm{e}^{2\pi\mathrm{i}\alpha\beta/q}W(\alpha) \tag{28}$$

where $\lambda_0 = \sum_{\alpha=0}^{q-1} W(\alpha)$. This result also applies to the $Z(q)$ model [5, 17]. For symmetric (and cyclic) $\mathbf{W}$ the transmissivities (28) are real if $W(\alpha)$ are real.
Cubic model. A discrete n-component cubic model [13] can be described by an interaction matrix with elements

$$W_{\alpha\beta} = \exp[K(\mathbf{S}_\alpha\cdot\mathbf{S}_\beta) + L(\mathbf{S}_\alpha\cdot\mathbf{S}_\beta)^2] \tag{29}$$

where $\mathbf{S}_\alpha$ is a vector which can point in one of the $2n$ directions along the positive and negative Cartesian axes in an n-dimensional space, i.e.

$$\mathbf{S}_\alpha = (\pm1,0,\ldots,0),(0,\pm1,\ldots,0),\ldots,(0,0,\ldots,\pm1). \tag{30}$$

This interaction matrix has three distinct eigenvalues

$$\begin{aligned} \lambda_0 &= \mathrm{e}^L(\mathrm{e}^K + \mathrm{e}^{-K}) + 2(n-1) & \\ \lambda_1 &= \mathrm{e}^L(\mathrm{e}^K - \mathrm{e}^{-K}) & n\text{-fold} \\ \lambda_2 &= \mathrm{e}^L(\mathrm{e}^K + \mathrm{e}^{-K}) - 2 & (n-1)\text{-fold} \end{aligned} \tag{31}$$

and hence we have

$$t_1 = \lambda_1/\lambda_0 \qquad t_2 = \lambda_2/\lambda_0. \tag{32}$$

These yield the known results [7].

Interaction model S_3-commuting subspace. In the preceding section we have considered one example of Biggs' interaction models corresponding to the group S_3, and showed that its interaction matrices commute in the subspace (27). Generally, the constraint (27) gives rise to intersections of two hypersurfaces in the parameter space. But there exist special solutions of (27) for which some of the Boltzmann weights are equal and the constraint is automatically satisfied. For positive Boltzmann weights there are three such cases, which are listed below together with the two additional transmissivities obtained from the further diagonalization of $\mathbf{T}_1$ and $\mathbf{T}_2$:

(i) $b = c, e = f, t_3 = (a - b + d - e)/\lambda_0, t_4 = (a - b - d + e)/\lambda_0$

(ii) $b = e = f, c = d, t_3 = (a - b)/\lambda_0, t_4 = (a - c)\lambda_0$

(iii) $b = d, c = e = f, t_3 = (a - b)/\lambda_0, t_4 = (a - c)\lambda_0$.

Case (ii) has also been noted very recently to be of interest in another context; its parameter space is naturally foliated in terms of elliptic curves [18].

Interaction model S_4. The group S_4 has 24 elements. Arranging them in the order of $\{\{S\},\{SP_{12}\}\}$, where $S \equiv \{e, P_{12}P_{34}, P_{13}P_{24}, P_{14}P_{23}, P_{24}P_{23}, P_{14}P_{12}, P_{12}P_{13}, P_{13}P_{14}, P_{23}P_{24}, P_{13}P_{12}, P_{14}P_{13}, P_{12}P_{14}\}$, we find the interaction matrix $\mathbf{W}$ to again assume the form (18) but now with

$$\mathbf{W}_1 = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{C} & \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{C} & \mathbf{A} \end{pmatrix} \qquad \mathbf{W}_2 = \begin{pmatrix} \mathbf{D}_{1234} & \mathbf{E}_{1234} & \mathbf{F}_{1234} \\ \mathbf{E}_{4321} & \mathbf{F}_{2143} & \mathbf{D}_{3412} \\ \mathbf{F}_{3412} & \mathbf{D}_{4321} & \mathbf{E}_{2143} \end{pmatrix} \tag{33}$$

where

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_3 & b_4 & b_1 & b_2 \\ b_4 & b_3 & b_2 & b_1 \\ b_2 & b_1 & b_4 & b_3 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_4 & c_3 & c_2 & c_1 \\ c_2 & c_1 & c_4 & c_3 \\ c_3 & c_4 & c_1 & c_2 \end{pmatrix}$$

$$\mathbf{D}_{ijkl} = \begin{pmatrix} d_i & d_j & d_k & d_l \\ d_j & d_i & d_l & d_k \\ d_l & d_k & d_j & d_i \\ d_k & d_l & d_i & d_j \end{pmatrix} \qquad \mathbf{E}_{ijkl} = \begin{pmatrix} e_i & e_j & e_k & e_l \\ e_k & e_l & e_i & e_j \\ e_j & e_i & e_l & e_k \\ e_l & e_k & e_j & e_i \end{pmatrix}$$

$$\mathbf{F}_{ijkl} = \begin{pmatrix} f_i & f_j & f_k & f_l \\ f_l & f_k & f_j & f_i \\ f_k & f_l & f_i & f_j \\ f_j & f_i & f_l & f_k \end{pmatrix}. \tag{34}$$

As before, this interaction matrix can be block-diagonalized by a similarity transformation generated by $\mathbf{P}$ given by (19), but now with

$$\mathbf{q} = \frac{1}{\sqrt{3}} \begin{pmatrix} \mathbf{I}_a & \mathbf{I}_a\mathbf{I}_b & \mathbf{I}_a\mathbf{I}_c \\ \mathbf{I} & \omega\mathbf{I}_b & \omega^2\mathbf{I}_c \\ \mathbf{I} & \omega^2\mathbf{I}_b & \omega\mathbf{I}_c \end{pmatrix} \qquad \mathbf{q}^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} \mathbf{I}_a & \mathbf{I} & \mathbf{I} \\ \mathbf{I}_b\mathbf{I}_a & \omega^2\mathbf{I}_b & \omega\mathbf{I}_b \\ \mathbf{I}_c\mathbf{I}_a & \omega\mathbf{I}_c & \omega^2\mathbf{I}_c \end{pmatrix} \tag{35}$$

where $\omega = \mathrm{e}^{\mathrm{i}2\pi/3}$, and

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{I}_a = \mathbf{I}_a^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\mathbf{I}_b = \mathbf{I}_b^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{I}_c = \mathbf{I}_c^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{36}$$

This again leads to the diagonal form (20) for $\mathbf{PWP}^{-1}$ with, in place of (22),

$$\mathbf{q}(\mathbf{W}_1 \pm \mathbf{W}_2)\mathbf{q}^{-1} = \begin{pmatrix} \lambda_\pm & 0 & 0 \\ 0 & \mathbf{m}_\pm & 0 \\ 0 & 0 & \mathbf{M}_\pm \end{pmatrix}$$

$$\lambda_\pm = \sum_{i=1}^{4} [a_i + b_i + c_i \pm (d_i + e_i + f_i)]$$

$$\mathbf{m}_\pm = \begin{pmatrix} (a \pm d)_{1234} & (b \pm e)_{1324} & (c \pm f)_{2314} \\ (c \mp e)_{1234} & (a \mp f)_{1324} & (b \mp d)_{2314} \\ (-b \pm f)_{1234} & (-c \pm d)_{1324} & (-a \pm e)_{2314} \end{pmatrix} \tag{37}$$

$$\mathbf{M}_\pm = \begin{pmatrix} \mathbf{A} + \omega^2\mathbf{B}\mathbf{I}_b + \omega\mathbf{C}\mathbf{I}_c & \pm[\mathbf{D}_{1234} + \omega\mathbf{E}_{1234}\mathbf{I}_b + \omega^2\mathbf{F}_{1234}\mathbf{I}_c] \\ \pm[\mathbf{D}_{1234} + \omega^2\mathbf{E}_{1234}\mathbf{I}_b + \omega\mathbf{F}_{1234}\mathbf{I}_c] & \mathbf{A} + \omega\mathbf{B}\mathbf{I}_b + \omega^2\mathbf{C}\mathbf{I}_c \end{pmatrix}$$

where

$$(a \pm d)_{ijkl} = (a_i + a_j - a_k - a_l) \pm (d_i + d_j - d_k - d_l) \tag{38}$$

etc. Thus, the transmissivity consists of $t = \lambda_-/\lambda_+$, the two 3×3 matrices $\mathbf{m}_\pm/\lambda_+$ and two 8×8 matrices $\mathbf{M}_\pm/\lambda_+$ given by (37) and (38).

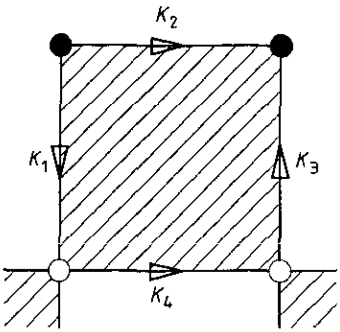

Figure 1. Unit cell of a chiral checkerboard lattice. The full circles denote spins that are decimated.

6. Disorder solution

Our formulation of the transmissivity permits a simple and direct derivation of disorder solutions [19]. Consider, for example, the case of a chiral square lattice a unit cell of which, say, the black square of a checkerboard, is shown in figure 1, where the lattice edges are oriented to indicate that the interaction matrix may not be symmetric. Define $W_{\alpha\beta}(\tilde{\mathbf{K}}) \equiv W_{\beta\alpha}(\mathbf{K})$ which is always possible, provided that the interaction is of the same class when the edge orientation is reversed. Then, in the example shown in figure 1, the criterion for disorder solution is [19]

$$\sum_{\beta,\gamma=0}^{q-1} W_{\alpha\beta}(\tilde{\mathbf{K}}_1)W_{\beta\gamma}(\mathbf{K}_2)W_{\gamma\delta}(\mathbf{K}_3) = \frac{\lambda}{W_{\alpha\delta}(\mathbf{K}_4)} \tag{39}$$

where λ is a multiplicative factor which turns out to be the per-site partition function. Combining (13) with (15) and using the fact that

$$\frac{1}{W_{\alpha\beta}(\mathbf{K})} = W_{\alpha\beta}(-\mathbf{K}) \tag{40}$$

we find from (39) the following conditions for the disorder solution:

$$t_i(\tilde{\mathbf{K}}_1)t_i(\mathbf{K}_2)t_i(\mathbf{K}_3) = t_i(-\mathbf{K}_4) \tag{41}$$

$$\mathbf{T}_i(\tilde{\mathbf{K}}_1)\mathbf{T}_i(\mathbf{K}_2)\mathbf{T}_i(\mathbf{K}_3) = \mathbf{T}_i(-\mathbf{K}_4) \tag{42}$$

where i ranges over all transmissivities. This results in the following expression for the per-site partition function:

$$\lambda = \lambda_0^3 / \sum_{\beta} W_{\alpha\beta}^{-1}. \tag{43}$$

7. Summary and discussions

We have shown that the transmissivity of a spin system, which are entities transforming according to the product property (2) under the decimation of spins, can be taken to be the eigenvalues as well as matrices which block-diagonalize the interaction matrices. If the interaction matrices with different parameters commute and are diagonalizable, then the transmissivity is given by the ratios of the eigenvalues. If the interaction matrices do not commute as in the example of the interaction model defined by S_3, the transmissivity entities are given by eigenvalues as well as block matrices. In all cases the largest eigenvalue is distinct permitting the use of $F_{\alpha\beta}$ given by (12) as a bond variable in diagrammatic expansions. Then we find all diagrams containing vertices with a single incident bond to vanish identically. In addition, diagrams transform according to (15) or (25) when spins are decimated.

Finally, it is useful to mention some possible extensions of our formulation. Our analysis of the transmissivity using the method of interaction matrices now opens the door for carrying out similar analyses for a host of other problems. These include the dual transmissivity [3], the break-collapse method widely used in real-space renormalization group studies [3, 7, 20], and the important connection of transmissivity with correlation functions [21, 22]. In addition, the new formulation of transmissivity as developed here can also be used as a tool to obtaining high-temperature expansions for complex discrete spin systems.

Acknowledgments

Two of us (JMM and FYW) would like to thank the Academia Sinica where this work was initiated for their hospitality. FYW is supported in part by National Science Foundation Grants DMR-9015489, INT-8902033 and INT-9113701; he is also grateful for the hospitality of the Laboratoire de Physique Théorique et Hautes Énergies where this work was completed. CKH is supported by the National Science Council of the Republic of China (Taiwan) under grant no NSC81-208-M-001-55. We would also like to thank the referee for calling our attention to [10], [20] and [21].

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